Moving Averages

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May 23, 2017

1 Moving Average or Running Mean

An algorithm for taking the moving average of the last j samples:

$$\bar{x}_n \leftarrow \bar{x}_{(n-1)} + \frac{x_n - x_{n-j}}{j} \tag{1}$$

The gain of the running mean varies with frequencies in the samples. The frequency response is a sinc function[1]. The EWMA however has constant gain with frequency.

2 Exponentially Weighted Moving Average

Given a series of samples: $x_1, x_2, \dots x_i, \dots x_n$ with x_n most recent, their discrete EWMA:

$$\bar{x}_n \equiv \frac{[e^{-1\beta}x_n + e^{-2\beta}x_{n-1} + \dots + e^{-(j+1)\beta}x_{n-j} + \dots + e^{-n\beta}x_1]}{[e^{-1\beta} + e^{-2\beta} + \dots + e^{-(j+1)\beta} + \dots + e^{-n\beta}]},$$

where higher β (or smaller time constant $1/\beta$) weights recent samples more strongly ($\beta > 0$).

Taking $\theta = e^{-\beta} \quad (0 < \theta < 1)$, gives

$$\begin{aligned}\bar{x}_n &\equiv \sum_{j=0}^{n-1} \theta^{j+1} x_{n-j} \Big/ \sum_{j=0}^{n-1} \theta^{j+1} \\ &\equiv \sum_{j=0}^{n-1} \theta^{j} x_{n-j} \Big/ \sum_{j=0}^{n-1} \theta^{j}. \qquad (2)\end{aligned}$$

The time constant $1/\beta = -1/\ln\theta$. The time constant of an EWMA can be thought of as how fast the EWMA responds to a step change in the inputs from one instant to another. It is the number of iterations needed to change the moving average by $1 - 1/e \approx 63\%$ of the step.

2.1 Iterative EWMA Algorithm

An iterative algorithm for this EWMA is

$$\begin{aligned}\bar{X}_n &\leftarrow (1-\theta)x_n + \theta\bar{X}_{n-1} \qquad (3)\\ &\leftarrow (1-\theta)x_n + \theta[(1-\theta)x_{n-1} + \theta\bar{X}_{n-2}] \\ &\leftarrow (1-\theta)[\theta^0 x_n + \theta^1 x_{n-1} + \dots + \theta^j x_{n-j} + \dots + \theta^{(n-1)}x_1] \\ &\leftarrow (1-\theta)\sum_{j=0}^{n-1}\theta^j x_{n-j}. \qquad (4)\end{aligned}$$

[1] http://www.numberwatch.co.uk/smoothing_of_data.htm

Comparing with Eqn 2

$$\bar{X}_n = (1-\theta)\left(\sum_{j=0}^{n-1}\theta^j\right)\bar{x}_n.$$

As $n \to \infty$, the Taylor series expansion of $1/(1-\theta)$ is $\sum_{j=0}^{n}\theta^j$, therefore,

$$\bar{X}_n \to \bar{x}_n.$$

2.2 Event-Based EWMA Algorithm

If samples do not arrive evenly, the above can be modified to an event-based EWMA. For instance, if the average is desired to move with time, then time can be slotted. Instead of updating the algorithm every time-slot, when a sample arrives n time-slots since the last sample, the EWMA can catch up by emulating n iterations of Eqn 3. If a new sample x_{n+1} arrives, the EWMA for the previous time slot $\bar{x}_n$ can be calculated assuming that $x_n = x_{n-1} = x_{n-2} = \ldots = x_1$ where x_1 was the previous sample and the EWMA prior to that was $\bar{x}_0$. From Eqn 3:

$$\begin{aligned}
\bar{x}_1 &\leftarrow (1-\theta)x_1 + \theta\bar{x}_0 \\
\bar{x}_2 &\leftarrow (1-\theta)(\theta^0 x_2 + \theta^1 x_1) + \theta^2\bar{x}_0 \\
\bar{x}_n &\leftarrow (1-\theta)(\theta^0 x_n + \theta^1 x_{n-1} + \theta^2 x_{n-2} + \ldots + \theta^{(n-1)})x_1 + \theta^n\bar{x}_0 \\
&\leftarrow (1-\theta)(\theta^0 + \theta^1 + \theta^2 + \ldots + \theta^{(n-1)})x_1 + \theta^n\bar{x}_0 \\
&\leftarrow (1-\theta^n)x_1 + \theta^n\bar{x}_0 \qquad (5)
\end{aligned}$$

2.3 EWMA Implementation

Alternative formulations of Eqns 3 and 5 with $\alpha = (1-\theta) \quad (0 < \alpha < 1)$ are:

$$\begin{aligned}
\bar{x}_n &\leftarrow \alpha x_n + (1-\alpha)\bar{x}_{n-1} \\
&\leftarrow \bar{x}_{n-1} + \alpha(x_n - \bar{x}_{n-1}) \qquad (6) \\
\bar{x}_n &\leftarrow (1-(1-\alpha)^n)x_1 + (1-\alpha)^n\bar{x}_0 \qquad (7) \\
&\neq \alpha^n x_1 + (1-\alpha)^n\bar{x}_0
\end{aligned}$$

Eqn 6 can be implemented on a binary machine with two adds and a shift if α is chosen to be a negative integer power of 2. Eqn 7 is not so easy to implement efficiently. However, if the arrival process of samples is Poisson, there is no need to make allowances for variations in the inter-arrivals of samples, which is the so-called PASTA property [Wol82].

References

[Wol82] Ronald W. Wolff. Poisson Arrivals See Time Averages. *Operations Research*, 30(2):223–231, March–April 1982.

Document history

Version	Date	Author	Details of change
00a	30-Dec-2005	Bob Briscoe	First Draft
01	23 May 2017	Bob Briscoe	Corrected Taylor series error.